

Nonuniqueness for specifications in $\ell^{2+\epsilon}$

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Abstract

For every $p > 2$, we construct a regular and continuous specification (g -function), which has a variation sequence that is in ℓ^p and which admits multiple Gibbs measures. Combined with a result of Johansson and Öberg, [12], this determines the optimal modulus of continuity for a specification which admits multiple Gibbs measures.

1 Introduction

For a finite set A let $P(A)$ be the set of probability distributions on A . A *specification* g (also commonly known as a g -function) is a measurable function from $A^{\mathbb{N}}$ to $P(A)$. A specification g is *regular* if there exists $\varrho > 0$ such that for every sequence $\{b_n\}_{n \in \mathbb{N}}$ and every $a \in A$ we have that $(g(\{b_n\}))(a) \geq \varrho$. We focus on specifications that are regular and continuous with respect to the product topology. A *Gibbs measure* for a specification g is a shift invariant probability measure μ on $A^{\mathbb{Z}}$ such that for every $f : A \rightarrow \mathbb{R}$,

$$\mathbf{E}_{\mu}(f(x_0)|x_{-1}, x_{-2}, \dots) = \mathbf{E}_{g(x_{-1}, x_{-2}, \dots)}(f) \quad \text{a.s.}$$

It is easy to show that every continuous specification has a Gibbs measure.

Given a past, x_{-1}, x_{-2}, \dots , the specification g tells us the probability distribution for x_0 , the next state of the process. Thus the specification and the past determine the stochastic evolution of the process. One common example of a specification is a finite state space Markov chain. The specification for a k -step Markov chain is determined by x_{-1}, \dots, x_{-k} . For

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this reason Döblin and Fortet referred to specifications as “chains with infinite connections” [6].

The question of whether a specification g uniquely determines the stationary process under a natural “mixing” assumption has been prominent since the pioneering work of Döblin and Fortet [6]. However, in the last three decades reasonable progress was achieved only in providing sufficient conditions for the uniqueness of Gibbs measure (see [20] for detailed discussions and references). Harris [11] studied the behavior of lumped Markov chains and introduced important coupling ideas that were used by several later authors. M. Keane [13] introduced the notion of a continuous g -function and gave conditions under which a g -function has a unique measure. One natural way to express uniqueness conditions is in terms of the modulus of continuity of g . To quantify the modulus of continuity of a specification g , we define the *variation of g at distance k* to be

$$\text{var}_k(g) = \sup \{ \|g(b) - g(b')\|_1 \mid b_1 = b'_1, b_2 = b'_2, \dots, b_k = b'_k \}.$$

The continuity of g is equivalent to the sequence $\text{var}_k(g) \rightarrow 0$.

In [21], Walters showed that if $\text{var}(g) \in \ell^1$, then the Gibbs measure is unique. Walters and Ledrappier [15] established a strong connection between specifications and the thermodynamic formalism of statistical mechanics. In particular they relate specifications with symbolic dynamics as developed in works of Sinai [18], Ruelle [17], Bowen [4], and others.

Walters’s work was sharpened by Lalley [14] and Berbee [3], who showed that the Gibbs measure is unique if

$$\sum_{n=1}^{\infty} \exp \left(- \sum_{m=1}^n \text{var}_m(g) \right) = \infty.$$

On the other hand, by the early 1980’s it was known that the equilibrium measures of appropriately chosen one dimensional long-range Ising models are not unique (see [7] for the hierarchical type models and [10] for the $1/r^2$ decaying cases). Existence of phase transition was later established for one-dimensional long-range percolation model (see [16] and [2]) and finally for one-dimensional FK-Random Cluster model [1]. Nevertheless a little more than a decade ago it was widely believed that continuous and regular specifications admit a unique Gibbs measure.

However, in 1993, Bramson and Kalikow [5] provided a remarkable (and until now unique) example of a continuous and regular specification that admits multiple Gibbs measures. The variation of the function g in Bramson and Kalikow’s construction is not in ℓ^p for any p . In

fact, in their example $\text{var}_k(g) \geq \frac{C}{\log k}$ for some constant C . This gave rise to the following question: For which values of p does $\text{var}(p) \in \ell^p$ imply uniqueness.

A few years ago Stenflo further sharpened Berbee's work [19]. However, the results of Berbee and Stenflo, while improving over Walters' result, are still in the realm of ℓ^1 . Recently Johansson and Öberg [12] showed that if g is regular and $\{\text{var}_k(g)\}_{k=1}^\infty$ is in ℓ^2 then g admits a unique Gibbs measure. Our main result is the following:

Theorem 1. *For every $p > 2$, there exists a regular specification g such that $\{\text{var}_k(g)\} \in \ell^p$ and g admits multiple Gibbs measures.*

This shows that the result of Öberg and Johansson is tight.

Remark. In [8, 9] Fernández and Maillard proved a Dobrushin type uniqueness condition. This condition is not comparable with the variation conditions.

2 Construction

We will use the alphabet of size four $A = \{+1, -1\}^2$. We fix a parameter $\epsilon \in (0, .5)$. Given this we pick a positive integer $K = K(\epsilon)$ such that the inequalities in lines (8) through (11) below are true for all $k \geq K$. We now begin to define a regular continuous specification $g(x, y) = g^\epsilon(x, y) : A^\mathbb{N} \rightarrow P(A)$.

The choice of (x_0, y_0) given $\{x_{-i}, y_{-i}\}_{i=1}^\infty$ consists of four steps:

1. Choose y_0 independently of $\{x_{-i}, y_{-i}\}_{i=1}^\infty$, so that $y_0 = +1$ with probability 0.5 and $y_0 = -1$ with probability 0.5.
2. Using the values of $\{y_{-i}\}_{i=0}^\infty$ choose a (deterministic) set of odd size $S \subset -\mathbb{N}$.
3. Using the values of $\{y_{-i}\}_{i=0}^\infty$ choose a (deterministic) value $0 \leq v < 0.4$.
4. Let z be the majority value of $\{x_t : t \in S\}$. Choose $x_0 = z$ with probability $0.5 + v$ and $x_0 = -z$ with probability $0.5 - v$.

In order to complete the first step the second coordinate (y) must be i.i.d. with distribution $(1/2, 1/2)$. We ensure this if for all x and y

$$g(x, y)(1, 1) + g(x, y)(-1, 1) = g(x, y)(1, -1) + g(x, y)(-1, -1) = \frac{1}{2}.$$

First we pick y_0 . We write \bar{y} to represent y and y_0 . The most intricate part of the construction is the choice of the set S . Before we choose S we need some notation.

For every positive integer k , let I_k be the sequence of length

$$\ell_k = \lceil (1 + \epsilon)^k \rceil$$

such that the last element is -1 and all other elements are $+1$. We also define

$$\beta_k = 2^{\ell_k} \tag{1}$$

and

$$\nu_k = \frac{\beta_k}{\beta_{k-1}} \sim 2^{\epsilon \cdot \ell_{k-1}} \tag{2}$$

Now we define a block structure which will allow us to choose the set $S \in -\mathbb{N}$ and a parameter v .

Definition 2. A complete k block in \bar{y} is a subsequence $\{y_i\}_{i=a}^{b-1}$ such that

1. $(\bar{y}_{a-\ell_k+1}, \bar{y}_{a-\ell_k+2}, \dots, \bar{y}_a) = (\bar{y}_{b-\ell_k+1}, \bar{y}_{b-\ell_k+2}, \dots, \bar{y}_b) = I_k$
2. for no $c \in (a, b)$ does $(\bar{y}_{c-\ell_k+1}, \bar{y}_{c-\ell_k+2}, \dots, \bar{y}_c) = I_k$

We also define a partial k block as follows.

Definition 3. A partial k block in \bar{y} is a subsequence $\{y_i\}_{i=a}^0$ such that

1. $(\bar{y}_{a-\ell_k+1}, \bar{y}_{a-\ell_k+2}, \dots, \bar{y}_a) = I_k$
2. for no $c > a$ does $(\bar{y}_{c-\ell_k+1}, \bar{y}_{c-\ell_k+2}, \dots, \bar{y}_c) = I_k$

Definition 4. A k block in \bar{y} is an interval that is either a complete k block or a partial k block.

If \bar{y} is well understood then we write $B = [a, b)$ to denote a k block $\{y_i\}_{i=a}^{b-1}$. Note that this definition is invariant under a shift of \bar{y} in the following sense: Given y and any integer $t < 0$ define $\sigma^t(\bar{y})$ by $\sigma^t(\bar{y})_i = y_{i+t}$ for all $i \leq 0$. Then for $b < 0$, if $[a, b)$ is a k block for $\sigma^t(\bar{y})$ then $[a+t, b+t)$ is a k block for \bar{y} . Also note that the length of a k block is likely to be close to β_k , and that the number of $k-1$ blocks inside a k block is likely to be close to ν_k . A precise statement of these claims will be used extensively in the next section.

Next we label all of the k blocks. Given a sequence \bar{y} and a positive integer k we will define $B_{k,i} = B_{k,i}(\bar{y})$ to be the i -th k block in \bar{y} . More precisely we define $a_{k,i} = a_{k,i}(\bar{y})$ and $b_{k,i} = b_{k,i}(\bar{y})$ such that

1. $[a_{k,i}, b_{k,i})$ is a k block in \bar{y} .
2. $b_{k,i+1} = a_{k,i}$ for all i
3. $b_{k,1} = 1$.

These sequences can be either finite or infinite. Then we define $B_{k,i} = [a_{k,i}, b_{k,i})$.

Given y and $k > K$ define $N_k(\bar{y})$ to be the number of $k-1$ blocks in the k block containing

0. More precisely, we take $N_k(\bar{y})$ to be so that

$$a_{k,1} = a_{k-1, N_k(\bar{y})}, \quad (3)$$

or $N_k = \infty$ if (3) has no solution. We also set $N_K(\bar{y}) = |a_{K,1}(\bar{y})|$.

Definition 5. We say that the **beginning** of a K block $B = [a, b)$ is the interval $O(B) = [a, \min(b, a + \beta_K^{1-\epsilon}))$.

For $k > K$ we say that the **beginning** of a k block $B = [a, b)$ is the $\nu_k^{1-\epsilon}$ first $k-1$ blocks in B , i.e.

$$O(B) = \begin{cases} [a, b) & \text{if } N_k(\sigma^b(\bar{y})) \leq \nu_k^{1-\epsilon} \\ [a, b_{k-1, N_k(\sigma^b(\bar{y})) - \nu_k^{1-\epsilon}}(\sigma^b(\bar{y}))) & \text{otherwise} \end{cases}.$$

Definition 6. The **opening** $C(B)$ of a k block B is the set of points $t \in B$ such that

1. t is in the beginning of its j -block for every $K \leq j \leq k$.
2. If $a_j(t)$ is the smallest element in the j -block containing t , then $t - a_j(t) < \beta_{j+1}$ for all $K \leq j \leq k$.

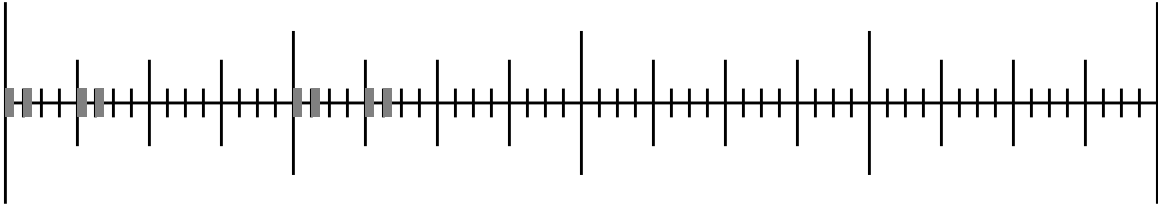


Figure 1: A block. The opening is marked in gray.

We also define C_k to be the union of $C(B)$ over all k blocks B . Note that if $t \in C_k$ then $t \in C_j$ for all $j < k$. The event that $0 \in C_k$ is determined by \bar{y}_i , $i \in [-\beta_{k+1}, 0]$. This fact

will be used to show that our specification is continuous and to show how quickly $\text{var}_k(g)$ approaches 0. Define k_0 to be the highest value such that $0 \in C_{k_0}$. If there is no such value then we take $k_0 = K - 1$. If $0 \in C_k$ for every k then we take $k_0 = \infty$.

We now define S to be the following set: If $k_0 = \infty$ then S is the empty set. If $|C(B_{k_0+1,1})|$ is odd then we take $S = C(B_{k_0+1,1})$. If $|C(B_{k_0+1,1})|$ is even, $S = C(B_{k_0+1,1}) \setminus \{\max(C(B_{k_0+1,1}))\}$.

Next we choose the value of v . If $\min(S) < -\beta_{k_0+2}$ then we take $v = 0$. Otherwise, we take

$$v = \begin{cases} 0.4 & \text{if } k_0 = K - 1 \\ \beta_{k_0}^{-\frac{1}{2} + \epsilon} & \text{if } k_0 \geq K. \end{cases} \quad (4)$$

Finally we set

$$g_1(x, y, y_0) = 0.5 + v \cdot \text{sign} \sum_{t \in S} x_t \quad (5)$$

where $\text{sign}(x)$ is the function that takes on values $+1$ or -1 depending on whether x is positive or negative. Thus we have defined our specification g by the coordinate functions

$$\begin{aligned} g_{(1,1)}(x, y) &= .5g_1(x, y, 1), \\ g_{(-1,1)}(x, y) &= .5(1 - g_1(x, y, 1)), \\ g_{(1,-1)}(x, y) &= .5g_1(x, y, -1), \\ g_{(-1,-1)}(x, y) &= .5(1 - g_1(x, y, -1)), \end{aligned}$$

We leave it to the reader to check the following lemma which says that g is symmetric in the x coordinate.

Lemma 1. *For all $x, y \in \{1, -1\}^{-\mathbb{N}}$ and $a, b \in \{1, -1\}$*

$$g_{(a,b)}(x, y) = g_{(-a,b)}(-x, y).$$

2.1 Continuity

We now show that g is continuous. The two most important elements of the construction are that for all y

1. there exists N such that the set S is determined by $\{y_i\}_{i=-N}^0$.
2. The larger N is the closer v is to zero.

In the next lemma we quantify these two statements and show that they imply that g is continuous.

Lemma 2. *The specification g is regular and continuous. Moreover the sequence $\text{var}_j(g) \in l^p$ for all*

$$p > \frac{2(1+\epsilon)^2}{1-2\epsilon}. \quad (6)$$

Note that as ϵ approaches 0, the bound in (6) goes to 2.

Proof. Let $k > K$. We want to estimate $\text{var}_j(g)$ for $\beta_k < j \leq \beta_{k+1}$.

Let $\{x_i^{(1)}, y_i^{(1)}\}_{i \in -\mathbb{N}}$ and $\{x_i^{(2)}, y_i^{(2)}\}_{i \in -\mathbb{N}}$ be such that $x_i^{(1)} = x_i^{(2)}$ and $y_i^{(1)} = y_i^{(2)}$ for every $i > -j$. It is enough to estimate $|g_1(x^{(1)}, y^{(1)}, l) - g_1(x^{(2)}, y^{(2)}, l)|$ for $l = -1, +1$. Fix l and for $h = 1, 2$ let

$$\bar{y}_i^{(h)} = \begin{cases} y_i^{(h)} & \text{if } i < 0 \\ l & \text{if } i = 0 \end{cases}.$$

If $k_0(\bar{y}^{(1)}) \leq k-2$ then $k_0(\bar{y}^{(1)}) = k_0(\bar{y}^{(2)})$ and g_1 depends only on $l, \{x_i^{(h)}, y_i^{(h)}\}_{i > -\beta_k}$. Therefore, on this case $g_1(x^{(1)}, y^{(1)}, l) - g_1(x^{(2)}, y^{(2)}, l) = 0$.

If, on the other hand, $k_0(\bar{y}^{(1)}) \geq k-1$, then $k_0(\bar{y}^{(2)}) \geq k-1$ as well and therefore $|g_1(x^{(h)}, y^{(h)}, l) - 0.5| < \beta_{k-1}^{-\frac{1}{2}+\epsilon}$. Thus

$$\text{var}_j(g) < 2(\beta_{k-1})^{-\frac{1}{2}+\epsilon} < 8 \left((\beta_{k+1})^{1/(1+\epsilon)^2} \right)^{-\frac{1}{2}+\epsilon} < 8j^{-\frac{1-2\epsilon}{2(1+\epsilon)^2}}. \quad (7)$$

□

3 Multiple measures

The goal of this section is to prove the following lemma:

Lemma 3. *For every $\epsilon \in (0, .5)$ the specification g^ϵ admits multiple measures.*

Theorem 1 is a direct consequence of Lemmas 2 and 3.

To see that the function g admits multiple Gibbs measures, we introduce the following notation. Choose an arbitrary t . We let $B_k(t)$ denote the k block containing t , and we let $C(B_k(t))$ denote the opening of $B_k(t)$. We will prove that $X(B_k(t)) = X(B_{k+1}(t))$ with extremely high probability. Next we note that for any t and t' that $B_k(t) = B_k(t')$ (and thus

$X(B_k(t)) = X(B_k(t'))$ for all k large enough. (See Figure 2 below.) Thus by the symmetry of Lemma 1 there exist at least two invariant measures: one where $\lim_{k \rightarrow \infty} X(B_k(t)) = 1$ a.s. and one where $\lim_{k \rightarrow \infty} X(B_k(t)) = -1$ a.s.

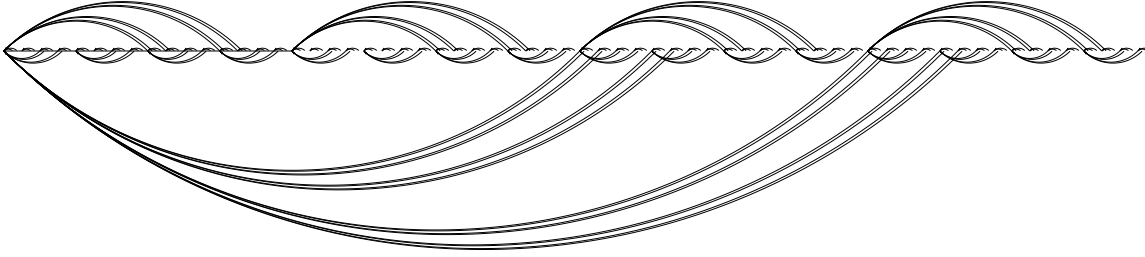


Figure 2: The arrows point from every point t to a point in $S(t)$.

Assume that μ is a stationary and ergodic measure on $\{(\pm 1, \pm 1)\}^{\mathbb{Z}}$ which is a Gibbs measure for g . Note that in the definitions of a complete k block (Definition 2 of Page 4) there was no requirement that a and b be negative. Thus we can use the same definition of a complete block for a sequence in $\{\pm 1\}^{\mathbb{Z}}$

Remark: From now on we will abbreviate the term **complete k block** to **k block**. This is a natural abuse of notation that results from the fact that we now speak about objects taking places in $\{(\pm 1, \pm 1)\}^{\mathbb{Z}}$ instead of $\{(\pm 1, \pm 1)\}^{-\mathbb{N}}$. In the same spirit, x and y will now denote two-sided sequences.

In subsections 3.1 and 3.2 we prove results about the block structure. The block structure depends only on the y sequence, which is an i.i.d. $(0.5, 0.5)$ sequence. In subsection 3.3 we use the results obtained in subsections 3.1 and 3.2 to understand the (more complicated) structure of the x sequence and show the existence of two different Gibbs measures.

3.1 Good blocks

In this subsection we define the notion of a good block. A good block is a block whose length is close to its expected length and the size of its opening is also close to its expected value. Our aim in this subsection is to show that a k block is good with high probability. In the next subsections we show that good blocks exhibit behavior that yields multiple Gibbs measures.

Definition 7. A k block B is **good** if

1. $|B| < \beta_k^{1+\epsilon}$ and
2. $|C(B)| > \beta_k^{1-\epsilon} 2^{-k}$

In order to show that most blocks are good we put a measure on k blocks by conditioning on y such that $\{y_i\}_{i=-l_k+2}^1 = I_k$. For a k block $B = \{y_i\}_{i=1}^{n-1}$ such that

1. $y_1 = -1$,
2. $y_{n-l_k} = \dots y_{n-1} = 1$ and
3. there is no occurrence of I_k in B

we define the measure

$$m_k(B) = \mathbf{P}(B_k(1) = B | \{y_i\}_{i=-l_k+2}^1 = I_k) = 2^{-|B|}.$$

For $t \in \mathbb{Z}$, we let $B_k(t)$ denote the k block containing t . With these definitions we can check that for any μ which is a Gibbs measure for g and any k and B

$$\mathbf{P}(B_k(0) \text{ is a translate of } B) = \frac{|B|m_k(B)}{\beta_k}.$$

Lemma 4. *For all $k \geq K$*

$$\sum_{B \text{ not good}} m_k(B) < 3 \cdot 2^{-3k}.$$

Proof. We prove the lemma by induction on k . For $k = K$ we need only to check the first condition. The probability that a k block is longer than $c\ell_k$ is less than $(1 - 2^{-\ell_k})^c$. Thus the probability that the first condition in the definition is not satisfied is at most

$$(1 - 2^{-\ell_k})^{\beta_k^{1+\epsilon}/2\ell_{k-1}} < 2^{-3k}. \quad (8)$$

This is true by our choice of K .

If $k > K$ and

1. $|B| < \beta_k^{1+\epsilon}$
2. there are at least $\nu_k^{1-\epsilon}$ $k-1$ blocks in B and
3. at least half of the $\nu_k^{1-\epsilon}$ $k-1$ blocks in the beginning of B are good

then

$$|C(B)| \geq \frac{1}{2} \nu_k^{1-\epsilon} \beta_{k-1}^{1-\epsilon} 2^{-k+1} \geq \beta_k^{1-\epsilon} 2^{-k} \quad (9)$$

and B is good.

As the bound in Line 8 holds for any $k > K$ we have that the probability that the first condition is not satisfied is at most 2^{-3k} . The probability that conditions 2 and 3 are satisfied is the probability that if we select $\nu_k^{1-\epsilon}$ $k-1$ blocks independently according to m that none of them are a k block and that at least half of them are good. The probability that a $k-1$ block is also a k block is ν_k^{-1} . Thus the probability of having a k block among $\nu_k^{1-\epsilon}$ $k-1$ blocks is less than

$$\nu_k^{1-\epsilon} \nu_k^{-1} < \nu_k^{-\epsilon} \leq 2^{-3k}. \quad (10)$$

By the induction hypothesis a $k-1$ block is not good with probability less than 2^{-k+1} . Thus the probability that half of a sequence of $\nu_k^{1-\epsilon}$ $k-1$ blocks chosen independently are not good is at most

$$2^{\nu_k^{1-\epsilon}} (2^{-k})^{.5\nu_k^{1-\epsilon}} \leq 2^{-\nu_k^{1-\epsilon}} \leq 2^{-3k}. \quad (11)$$

□

Lemma 5. *For every $k \geq K$*

$$\mathbf{P}(B_k(0) \text{ is good}) > 1 - 2^{-k}.$$

Proof. First, we want to estimate

$$\tilde{Y} = \sum_{|B| \leq \beta_k} m_k(B).$$

\tilde{Y} is the probability that I_k appears somewhere in $\{y_j\}_{j=1}^{\beta_k}$. For $i = 1, \dots, \ell_k$, let E_i be the event that there exists h with $h \equiv i \pmod{\ell_k}$ such that $(y_h, y_{h+1}, \dots, y_{h+\ell_k-1}) = I_k$. $\{E_i\}$ are negatively associated, and for every i ,

$$\mathbf{P}(E_i) = 1 - (1 - \beta_k^{-1})^{\beta_k/\ell_k}.$$

Therefore

$$\tilde{Y} \geq 1 - (1 - \beta_k^{-1})^{\beta_k}. \quad (12)$$

On the other hand, for $i = 1, \dots, 1000$ let F_i to be the event that I_k appears in

$$\{y_j : j = \beta_k i / 1000 \dots \beta_k (i + 1) / 1000 - 1\}.$$

Then the F_i -s are independent and using a first-moment argument we see that for every i , $\mathbf{P}(F_i) \leq 1/1000$. The event measured by \tilde{Y} is the union of the F_i -s plus the event that I_k appears on the seams between the blocks. Therefore,

$$\tilde{Y} \leq 1 - (1 - 1/1000)^{1000} + 1000\ell_k 2^{-\ell_k}. \quad (13)$$

For any j , let

$$\tilde{Z} = \sum_{j\beta_k < |B| \leq (j+1)\beta_k} m_k(B)$$

be the probability that the first appearance of I_k is in the interval $[j\beta_k, (j+1)\beta_k)$. Then

$$\tilde{Z} = \tilde{Y} \left(1 - \tilde{Y}\right)^j \left(1 - \ell_k 2^{-\ell_k}\right)^j, \quad (14)$$

where the last term comes from the event that I_k appears on the seam between two consecutive intervals of length β_k .

From (12), (13) and (14), we get that for any j we have that

$$(5/2)^{-j} > \sum_{j\beta_k < |B| \leq (j+1)\beta_k} m_k(B) > 2^{-2j-2}.$$

From this we get

$$\sum_{|B| > k\beta_k} m_k(B) > \sum_{j \geq k} 2^{-2j-2} > 3 \cdot 2^{-3k}$$

and

$$\sum_{|B| > k\beta_k} |B| m_k(B) < \sum_{j \geq k} (5/2)^{-j} (j+1) \beta_k < 2^{-k} \beta_k.$$

Thus for any set S of k blocks such that

$$\sum_{B \in S} m_k(B) < 3 \cdot 2^{-3k}$$

we have that

$$\sum_{B \in S} |B| m_k(B) < 2^{-k} \beta_k.$$

Combining this last statement with Lemma 4

$$\mathbf{P}(B_k(0) \text{ is good}) \geq 1 - \frac{1}{\beta_k} \left(\sum_{B \text{ not good}} |B| m_k(B) \right) > 1 - 2^{-k}.$$

□

3.2 Beautiful points

In this subsection we define the notion of a beautiful point. Our goal in this subsection will be to show that most points are beautiful.

Definition 8. We say that $t \in \mathbb{Z}$ is k **beautiful** if for every $j \geq k$,

1. $B_j(t)$ is good and
2. t is not in the beginning of $B_j(t)$. (The beginning of a block was defined in Definition 5 of page 5)

In this subsection we state two easy lemmas:

Lemma 6. If t and s belong to the same k block, and t is $k+1$ beautiful, then s is also $k+1$ beautiful.

Proof. Lemma 6 follows immediately from Definition 8. □

Lemma 7. Almost surely, for every t there exists $\hat{k}(t)$ such that t is $\hat{k}(t)$ beautiful.

Proof. First we show that

$$\mathbf{P}(0 \text{ is } k \text{ beautiful}) > 1 - 2^{-k+2}.$$

Lemma 5 tells us that the probability that the j block containing 0 is good is greater than $1 - 2^{-j}$. As there are at most $\nu_j^{1-\epsilon} k - 1$ blocks in the beginning of $B_k(0)$ and the expected number of $k - 1$ blocks in $B_k(0)$ is ν_j , the probability that 0 is in the beginning of $B_j(0)$ is less than

$$\nu_j^{1-\epsilon} / \nu_j = \nu_j^{-\epsilon}.$$

Therefore 0 is k beautiful with probability at least

$$1 - \sum_{j=k}^{\infty} (2^{-j} + \nu_j^{-\epsilon}) = 1 - \sum_{j=k}^{\infty} 2^{-j+1} = 1 - 2^{-k+2}.$$

Thus by Borel-Cantelli we get that $\hat{k}(0)$ exists a.s. The lemma is true because μ is shift invariant. □

3.3 Proof of lemma 3

In the previous subsections we only discussed the structure induced by the y values. In this subsection we will shed some light on the structure of the x values. For a k block B , we define the **signature** of B to be

$$X(B) = \text{sign} \sum_{t \in C'(B)} x_t \quad (15)$$

where

$$C'(B) = \begin{cases} C(B) & \text{if } |C(B)| \text{ is odd} \\ C(B) \setminus \{\max(C(B))\} & \text{otherwise} \end{cases}.$$

Note that since $C'(B)$ is always odd, $X(B)$ can only be $+1$ or -1 . We assume that μ is an ergodic Gibbs measure for g .

Lemma 8. *For all t*

$$X(t) = \lim_{k \rightarrow \infty} X(B_k(t))$$

exists a.s. and is equal to 1 or -1 .

Proof. By Lemma 7 we get \hat{k} such that t is \hat{k} beautiful. By Lemma 1 it causes no loss of generality to assume that $X(B_{k+1}(t)) = 1$. Using that $B_k(t) = [a, b)$ is good for every $k > \hat{k}$, that μ is a Gibbs measure with respect to g , and line (4) in the definition of g , we get that given y and $\{x_i\}_{i < a}$ the values of $x_s : s \in C'(B_k(t))$ are independent and identically distributed. For every s in $C'(B_k(t))$,

$$\mathbf{P}(x_s = 1 \mid y, \{x_i\}_{i < a}) = \frac{1}{2} + \beta_k^{-\frac{1}{2} + \epsilon}. \quad (16)$$

Also, the fact that $B_k(t)$ is good implies

$$|C'(B_k(t))| \geq \beta_k^{1-\epsilon} 2^{-k} - 1.$$

Thus we get

$$\mathbf{P}(X(B_k(t)) = X(B_{k+1}(t))) = \mathbf{P}\left(\sum_{s \in C'(B_k(t))} x_s > 0 \mid y, \{x_i\}_{i < a}\right).$$

We have that

$$\mathbf{E}\left(\sum_{s \in C'(B_k(t))} x_s \mid y, \{x_i\}_{i < a}\right) = 2|C'(B_k(t))|\beta_k^{-\frac{1}{2} + \epsilon}.$$

The standard deviation of the sum is less than $\frac{1}{2}\sqrt{|C'(B_k(t))|}$. Thus by Markov's inequality we have that

$$\begin{aligned}
\mathbf{P}(X(B_k(t)) = X(B_{k+1}(t))) &\leq \mathbf{P}\left(\sum_{s \in C'(B_k(t))} x_s > 0 \mid y, \{x_i\}_{i < a}\right) \\
&\leq \left(\frac{2|C'(B_k(t))|\beta_k^{-\frac{1}{2}+\epsilon}}{\frac{1}{2}\sqrt{|C'(B_k(t))|}}\right)^{-2} \\
&\leq 16|C'(B_k(t))|^{-1}\beta_k^{1-2\epsilon} \\
&\leq 32\beta_k^{1-\epsilon}2^k\beta_k^{1-2\epsilon} \\
&\leq 32\beta_k^{-\epsilon}2^k \\
&\leq 2^{-k}.
\end{aligned}$$

By Borel-Cantelli there are only finitely many values of k such that $X(B_k(t)) \neq X(B_{k+1}(t))$ and $X(t)$ exists. As $|C'(B_k(t))|$ is odd for all k and t the limit must be either 1 or -1 . \square

We are now ready to prove Lemma 3.

Proof of Lemma 3. For every t and s $B_k(t) = B_k(s)$ for all k sufficiently large. Therefore $X(t) = X(s)$ for all s and t . (See Figure 2 at page 8.) Since μ is ergodic and $X(0)$ is shift invariant, $X(0)$ is a μ almost sure constant, which we denote by $X(\mu)$. Let

$$\tilde{\mu}(A) = \mu(\{(x, y) : (-x, y) \in A\}).$$

Then, by Lemma 1, $\tilde{\mu}$ is also a Gibbs measure for g . On the other hand, $X(\tilde{\mu}) = -X(\mu) \neq X(\mu)$, and therefore $\mu \neq \tilde{\mu}$. \square

4 An open problem

Our construction is not monotone - by changing a y value from a -1 to a $+1$ we may change the set S at which we look, and then reduce the probability that the function outputs $+1$ at the x coordinate. The construction of Bramson and Kalikow, on the other hand, is monotone. Therefore we ask the following question:

Question 1. *Is there a value of p and a continuous monotone regular specification g such that $\text{var}(g) \in \ell^p$ and g admits multiple Gibbs measures?*

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